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Landau solutions for incompressible Navier-Stokes equations and applications

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Dedicated to Professor Kenji Nishihara on the occasion of his 60th birthday

1 Introduction

This article is based on a joint work with Tai-Peng Tsai (University of British Columbia). We consider point singularities of very weak solutions of the 3D stationary Navier-Stokes equations in a finite region Ω in $\mathbb{R}^3$. The Navier-Stokes equations for the velocity $u:\Omega\rightarrow \mathbb{R}^3$ and pressure $p:\Omega\rightarrow \mathbb{R}$ with external force $f:\Omega\rightarrow \mathbb{R}^3$ are

$$-\Delta u+(u\cdot\nabla)u+\nabla p=f, \quad \operatorname{div} u=0, \quad (x\in\Omega). \tag{1.1}$$

A *very weak solution* is a vector function u in $L_{loc}^{2}(\Omega)$ which satisfies (1.1) in distribution sense:

$$\int -u\cdot\Delta\varphi+u_{j}u_{i}\partial_{j}\varphi_{i}=\langle f,\varphi\rangle, \quad \forall\varphi\in C_{c,\sigma}^{\infty}(\Omega),$$

and $\int u\cdot\nabla h=0$ for any $h\in C_{c}^{\infty}(\Omega)$. Here the force f is allowed to be a distribution and

$$C_{c,\sigma}^{\infty}(\Omega)=\{\varphi\in C_{c}^{\infty}(\Omega, \mathbb{R}^{3}) : \operatorname{div}\varphi=0\}.$$

In this definition the pressure is not needed. Denote $B_{R}=\{x\in \mathbb{R}^{3} : |x|<R\}$ and $B_{R}^{c}=\mathbb{R}^{3}\setminus B_{R}$ for $R>0$.

We are concerned with the behavior of very weak solutions which solve (1.1) in the punctured ball $B_{2}\backslash\{0\}$ with zero force, i.e., $f=0$. There are a lot of studies on this problem [3, 11, 13, 14, 2, 8]. Shapiro [13, 14] proved the removable singularity theorem under some assumptions on u. He proved that if $u\in L^{3+\varepsilon}(B_{2})$ for some $\varepsilon>0$ and $u(x)=o(|x|^{-1})$ $(x\rightarrow 0)$, then (u,p) can be defined at 0 so that it is a smooth solution of (1.1) in the whole ball B_2. Choe and Kim [2] obtained similar results by using the theories of the hydrodynamic potentials and homogeneous harmonic polynomials. Kim and Kozono [8] recently proved that if $u \in L^3(B_2)$ or $u(x) = o(|x|^{-1})$ $(x \to 0)$, then the same conclusion holds. As mentioned in [8], their result is optimal in the sense that if their assumption is replaced by

$$|u(x)| \le C_* |x|^{-1} \tag{1.2}$$

for $0 < |x| < 2$, then the singularity is not removable in general, due to *Landau solutions*, which is the family of explicit singular solutions calculated by L. D. Landau [6].

The purpose of this article is to characterize the singularity and to identify the leading order behavior of very weak solutions satisfying the threshold assumption (1.2) when the constant C_* is sufficiently small. We show that it is given by Landau solutions. In order to state main result, we recall Landau solutions.

Landau obtained his solutions in 1944, see [6, 7]. They can be parametrized by vectors $b \in \mathbb{R}^3$ in the following way: For each $b \in \mathbb{R}^3$ there exists a unique (-1)-homogeneous solution U^b of (1.1) together with an associated pressure P^b which is (-2)-homogeneous, such that U^b, P^b are smooth in $\mathbb{R}^3 \backslash \{0\}$ and they solve

$$-\Delta u + (u \cdot \nabla) u + \nabla p = b\delta, \quad \operatorname{div} u = 0. \tag{1.3}$$

in $\mathbb{R}^3$ in the sense of distributions, where δ denotes the Dirac δ function. When $b = (0, 0, \beta)$, they have the following explicit formulas in spherical coordinates r, θ, ϕ with $x = (r \sin\theta \cos\phi,\ r \sin\theta \sin\phi,\ r\cos\theta)$:

$$U = \frac{2}{r}\left(\frac{A^2 - 1}{(A - \cos\theta)^2} - 1\right) e_r - \frac{2\sin\theta}{r(A - \cos\theta)} e_\theta, \quad P = \frac{-4(A\cos\theta - 1)}{r^2(A - \cos\theta)^2} \tag{1.4}$$

where $e_r = \frac{x}{r}$ and $e_\theta = (-\sin\theta\sin\phi,\ \sin\theta\cos\phi,\ \cos\theta)$. The parameters $\beta \ge 0$ and $A \in (1, \infty]$ are related by the formula

$$\beta = 16\pi \left(A + \frac{1}{2}A^2 \log\frac{A-1}{A+1} + \frac{4A}{3(A^2 - 1)}\right).$$

The formulas for general b can be obtained from rotation. One checks directly that $\|rU^b\|_{L^\infty}$ is monotone in $|b|$ and $\|rU^b\|_{L^\infty} \to 0$ (or ∞) as $|b| \to 0$ (or ∞). Recently Sverak [15] observed that Landau solutions were the only solutions of (1.1) in $\mathbb{R}^3 \backslash \{0\}$ which are smooth and (-1)-homogeneous in $\mathbb{R}^3 \backslash \{0\}$, without assuming axisymmetry. Hence Landau solutions can be regarded as the canonical family of the solutions for (1.1). See also [18, 1, 9] for related results.

If u, p is a solution of (1.1), we will denote by

$$T_{ij}(u, p) = p\delta_{ij} + u_i u_j - \partial_i u_j - \partial_j u_i$$

the momentum flux density tensor in the fluid, which plays an important role to determine the equation for (u, p) at 0. Our main result is the following.

Theorem 1.1 *For any $q \in (1,3)$, there is a small $C_* = C_*(q) > 0$ such that, if u is a very weak solution of* (1.1) *with zero force in $B_2 \backslash \{0\}$ satisfying* (1.2) *in $B_2 \backslash \{0\}$, then there is a scalar function p satisfying $|p(x)| \le C|x|^{-2}$, unique up to a constant, so that (u,p) satisfies* (1.3) *in B_2 with $b_i = \int_{|x|=1} T_{ij}(u,p) n_j(x)$, and*

$$\left\| u - U^b \right\|_{W^{1,q}(B_1)} + \sup_{x \in B_1} |x|^{3/q-1} |(u - U^b)(x)| \le C C_*, \tag{1.5}$$

where the constant C is independent of q and u.

The exponent q can be regarded as the degree of the approximation of u by U^b. The closer q gets to 3, the less singular $u - U^b$ is. But in our theorem, $C_*(q)$ shrinks to zero as $q \to 3_-$. Ideally, one would like to prove that $u - U^b \in L^\infty$. However, it seems quite subtle in view of the following model equation for a scalar function,

$$-\Delta v + cv = 0, \quad c = \Delta v / v.$$

If we choose $v = \log |x|$, then $c(x) \in L^{3/2}$ and $\lim_{|x| \to 0} |x|^2 |c(x)| = 0$, but $v \notin L^\infty$. In equation (3.2) for the difference $w = u - U^b$, there is a term $(w \cdot \nabla) U^b$ which has similar behavior as cv above.

This work is inspired by Korolev-Sverak [9] in which they study the asymptotic as $|x| \to \infty$ of solutions of (1.1) satisfying (1.2) in $\mathbb{R}^3 \setminus B_1$. They show that the leading behavior is also given by Landau solutions if C_* is sufficiently small. Our theorem can be considered as a dual version of their result. However, their proof is based on the unique existence of the difference $\varphi(u - U^b)$ where φ is a cut-off function supported near infinity. If one tries the same approach for our problem, one needs to choose a sequence φ_k with the supports of $1 - \varphi_k$ shrinking to the origin, which produce very singular force terms near the origin. Instead, we prove Lemma 2.3 which defines the equation for (u,p) at the origin. Since the equation for u is same as U^b near the origin, the δ-functions at the origin cancel in the equation for the difference. Then applying the approach of Kim-Kozono [8], we prove the unique existence of the difference in $W_0^{1,r}(B_2)$ for $3/2 \le r < 3$ and uniqueness in $W_0^{1,r} \cap L^3_{wk}(B_2)$ for $1 < r < 3/2$, where $W_0^{1,r}(B_2)$ is the closure of $C_0^\infty(B_2)$ in the norm $W_0^{1,r}(B_2)$.

2 Preliminaries

In this section we collect some lemmas for the proof of Theorem 1.1. The first lemma recalls O'Neil's inequalities [12], which are Hölder type inequalities in Lorentz spaces.

See [10, 8] for simpler proofs in these special cases. We denote the Lorentz spaces by $L^{p,q}$ $(1 < p < \infty,\ 1 \le q \le \infty)$.

Note $L^3_{wk} = L^{3,\infty}$.

Lemma 2.1 *Let $B = B_2 \subset \mathbb{R}^n$, $n \geq 2$.*

i) Let $1 < p_1, p_2 < \infty$ with $1/p := 1/p_1 + 1/p_2 < 1$ and let $1 \le r_1, r_2 \le \infty$. For $f \in L^{p_1,r_1}$ and $g \in L^{p_2,r_2}$, we have

$$\|fg\|_{L^{p,r}(B)} \le C\|f\|_{L^{p_1,r_1}(B)}\|g\|_{L^{r_1,r_2}(B)} \qquad \text{for } r := \min\{r_1, r_2\},$$

where $C = C(p_1, r_1, p_2, r_2)$.

ii) Let $1 < r < n$. For $f \in W^{1,r}(B)$, we have

$$\|f\|_{L^{\frac{nr}{n-r},r}(B)} \le C\|f\|_{W^{1,r}(B)},$$

where $C = C(n, r)$.

For our application, we will let $n = 3$, $1 < r < 3$, and we have

$$\|fg\|_{L^r(B)} \le C\,\|f\|_{L^3_{wk}}\,\|g\|_{L^{\frac{3r}{3-r},r}} \le C_r\,\|f\|_{L^3_{wk}(B)}\,\|g\|_{W^{1,r}(B)}\,. \tag{2.1}$$

The next lemma is on interior estimates for Stokes system with no assumption on the pressure.

Lemma 2.2 *Assume $v \in L^1$ is a distribution solution of the Stokes system*

$$-\Delta v_i + \partial_i p = \partial_j f_{ij}, \quad \operatorname{div} v = 0 \quad \text{in } B_{2R}$$

and $f \in L^r$ for some $r \in (1,\infty)$. Then $v \in W^{1,r}_{loc}$ and, for some constant C_r independent of v and R,

$$\|\nabla v\|_{L^r(B_R)} \le C_r\,\|f\|_{L^r(B_{2R})} + C_r R^{-4+3/r}\,\|v\|_{L^1(B_{2R})}\,.$$

This lemma is [17], Theorem 2.2. Although the statement in [17] assumes $v \in W^{1,r}_{loc}$, its proof only requires $v \in L^1$. This lemma can be also considered as [?, Lemma A.2] restricted to time-independent functions.

The following lemma shows the first part of Theorem 1.1, except (1.5). In particular, it shows that (u,p) solves (1.3).

Lemma 2.3 *If u is a very weak solution of (1.1) with zero force in $B_2\backslash\{0\}$ satisfying (1.2) in $B_2\backslash\{0\}$ (with C_* allowed to be large), there is a scalar function p satisfying $|p(x)| \le C|x|^{-2}$, unique up to a constant, such that (u,p) satisfies (1.3) in B_2 with $b_i = \int_{|x|=1} T_{ij}(u,p)n_j(x)$. Moreover, u, p are smooth in $B_2\backslash\{0\}$.*

Proof. For each $R \in (0, 1/2]$, u is a very weak solution in $B_2 - \bar{B}_R$ in L^∞. Lemma 2.2 shows u is a weak solution in $W^{1,2}_{loc}$. The usual theory shows that u is smooth and there is a scalar function p_R, unique up to a constant, so that (u, p_R) solves (1.1) in $B_2 - \bar{B}_R$, see e.g. [5]. By the scaling argument in Sverak-Tsai [17] using Lemma 2.2, we have for $x \in B_{3R} - B_{2R}$,

$$|\nabla^k u(x)| \le \frac{C_k C_*}{|x|^{k+1}} \qquad \text{for } k = 1, 2, \ldots, \tag{2.2}$$

where $C_k = C_k(C_*)$ are independent of $R \in (0, 1/2]$ and its dependence on C_* can be dropped if $C_* \in (0, 1)$. Varying R, (2.2) is valid for $x \in B_{3/2}\backslash\{0\}$. Since p_R is unique up to a constant, we can fix it by requiring $p_R = p_{1/2}$ in $B_2 \backslash \bar{B}_{1/2}$, and define $p(x) = p_R(x)$ for any $x \in B_2\backslash\{0\}$ with $R = |x|/2$. By the equation, $|\nabla p(x)| \le CC_*|x|^{-3}$. Integrating from $|x| = 1$ we get $|p(x)| \le CC_*|x|^{-2}$. In particular

$$|T_{ij}(u, p)(x)| \le CC_*|x|^{-2} \qquad \text{for } x \in B_{3/2}\backslash\{0\}. \tag{2.3}$$

Denote $NS(u) = -\Delta u + (u \cdot \nabla)u + \nabla p$. We have $NS(u)_i = \partial_j T_{ij}(u)$ in the sense of distributions. Thus, by divergence theorem and $NS(u) = 0$ in $B_2\backslash\{0\}$,

$$b_i = \int_{|x|=1} T_{ij}(u, p) n_j(x) = \int_{|x|=R} T_{ij}(u, p) n_j(x) \tag{2.4}$$

for any $R \in (0, 2)$. Let ϕ be any test function in $C_c^\infty(B_1)$. For small $\varepsilon > 0$,

$$\begin{aligned}
\langle NS(u)_i, \phi\rangle &= -\int T_{ij}(u)\partial_j\phi \\
&= -\int_{B_1\backslash B_\varepsilon} T_{ij}(u)\partial_j\phi - \int_{B_\varepsilon} T_{ij}(u)\partial_j\phi \\
&= \int_{B_1\backslash B_\varepsilon} \partial_j T_{ij}(u)\phi + \int_{\partial B_\varepsilon} T_{ij}(u)\phi n_j - \int_{\partial B_1} T_{ij}(u)\phi n_j - \int_{B_\varepsilon} T_{ij}(u)\partial_j\phi.
\end{aligned}$$

In the last line, the first integral is zero since $NS(u) = 0$ and the third integral is zero since $\phi = 0$. By the pointwise estimate (2.3), the last integral is bounded by $C\varepsilon^{3-2}$. On the other hand, by (2.4),

$$\int_{\partial B_\varepsilon} T_{ij}(u)\phi n_j \to b_i\phi(0) \quad \text{as} \quad \varepsilon \to 0.$$

Thus (u, p) solves (1.3) and we have proved the lemma. □

It follows from the proof that $|b| \le CC_*$ for $C_* < 1$. With this lemma, we have completely proved Theorem 1.1 in the case $q < 3/2$. In the case $3/2 \le q < 3$, it remains to prove (1.5).

3 Proof of main theorem

In this section, we present the proof of Theorem 1.1. We first prove that solutions belong to $W^{1,q}$. We next apply this result to obtain the pointwise estimate. For what follows, denote

$$w = u - U, \quad U = U^b. \tag{3.1}$$

By Lemma 2.3, there is a function $\tilde{p}$ such that $(w, \tilde{p})$ satisfies in B_2 that

$$\begin{gathered} -\Delta w + U \cdot \nabla w + w \cdot \nabla(U + w) + \nabla \tilde{p} = 0, \quad \operatorname{div} w = 0, \\ |w(x)| \le \frac{CC_*}{|x|}, \quad |\tilde{p}(x)| \le \frac{CC_*}{|x|^2}. \end{gathered} \tag{3.2}$$

Note that the δ-functions at the origin cancel.

3.1 $W^{1,q}$ regularity

In this subsection we will show $w \in W^{1,q}(B_1)$. Fix a cut off function φ with $\varphi = 1$ in $B_{9/8}$ and $\varphi = 0$ in $B^c_{11/8}$. We localize w by introducing

$$v = \varphi w + \zeta$$

where ζ is a solution of the problem $\operatorname{div} \zeta = -\nabla\varphi \cdot w$. By Galdi [4, Ch.3] Theorem 3.1, there exists such a ζ satisfying

$$\operatorname{supp} \zeta \subset B_{3/2} \backslash B_1, \quad \|\nabla \zeta\|_{L^{100}} \le C\|\nabla\varphi \cdot w\|_{L^{100}} \le CC_*.$$

The vector v is supported in $\bar{B}_{3/2}$ and satisfies $v \in W^{1,r} \cap L^3_{wk}$ for $r < 3/2$,

$$-\Delta v + U \cdot \nabla v + v \cdot \nabla(U + v) + \nabla \pi = f, \quad \operatorname{div} v = 0, \tag{3.3}$$

where $\pi = \varphi \tilde{p}$,

$$\begin{aligned} f = -2(\nabla\varphi \cdot \nabla)w - (\Delta\varphi)w + (U \cdot \nabla\varphi)w + (\varphi^2 - \varphi)w \cdot \nabla w + (w \cdot \nabla\varphi)w + \tilde{p}\nabla\varphi \\ -\Delta\zeta + (U \cdot \nabla)\zeta + \zeta \cdot \nabla(U + \varphi w + \zeta) + \varphi w \cdot \nabla\zeta \end{aligned}$$

is supported in the annulus $\bar{B}_{3/2} \backslash B_1$. One verifies directly that, for some C_1,

$$\sup_{1 \le r \le 100} \|f\|_{W_0^{-1,r}(B_2)} \le C_1 C_*. \tag{3.4}$$

Our proof is based on the following lemmas.

Lemma 3.1 (Unique existence) *For any* $3/2 \le r < 3$, *for sufficiently small* $C_* = C_*(r) > 0$, *there is a unique solution* v *of* (3.3)–(3.4) *in the set*

$$V = \{v \in W_0^{1,r}(B_2), \quad \|v\|_V := \|v\|_{W_0^{1,r}(B_2)} \le C_2 C_*\}$$

for some $C_2 > 0$ *independent of* $r \in [3/2, 3)$.

Lemma 3.2 (Uniqueness) *Let* $1 < r < 3/2$. *If both* v_1 *and* v_2 *are solutions of* (3.3)–(3.4) *in* $W_0^{1,r} \cap L^3_{wk}$ *and* $C_* + \|v_1\|_{L^3_{wk}} + \|v_2\|_{L^3_{wk}}$ *is sufficiently small, then* $v_1 = v_2$.

Assuming the above lemmas, we get $W^{1,q}$ regularity as follows. First we have a solution $\tilde v$ of (3.3) in $W_0^{1,q}(B_2)$ by Lemma 3.1. On the other hand, both $v = \varphi w + \zeta$ and $\tilde v$ are small solutions of (3.3) in $W_0^{1,r} \cap L^3_{wk}(B_2)$ for $r = 5/4$, and thus $v = \tilde v$ by Lemma 3.2. Thus $v \in W_0^{1,q}(B_2)$ and $w \in W^{1,q}(B_1)$.

Proof of Lemma 3.1. Consider the following mapping Φ: For each $v \in V$, let $\bar v = \Phi v$ be the unique solution in $W_0^{1,r}(B_2)$ of the Stokes system

$$\begin{aligned} -\Delta \bar v + \nabla \bar\pi &= f - \nabla \cdot (U \otimes v + v \otimes (U + v)) \\ \operatorname{div} \bar v &= 0. \end{aligned}$$

By estimates for the Stokes system, see Galdi [4, Ch.4] Theorem 6.1, in particular (6.9), for $1 < r < 3$, we have

$$\begin{aligned} \|\bar v\|_{W_0^{1,r}(B_2)} &\le C_r \|f\|_{W_0^{-1,r}} + C_r \|\nabla \cdot (U \otimes v + v \otimes (U + v))\|_{W_0^{-1,r}} \\ &\le C_r C_1 C_* + C_r \|U \otimes v + v \otimes (U + v)\|_{L^r}. \end{aligned}$$

By Lemma 2.1, in particular (2.1), for $1 < r < 3$,

$$\|\bar v\|_{W_0^{1,r}(B_2)} \le C_r C_1 C_* + C_r \tilde C_r (\|U\|_{L^3_{wk}} + \|v\|_{L^3_{wk}}) \|v\|_V.$$

We now choose $C_2 = 2C_r C_1$. Since $V \subset L^3_{wk}$ if $r \ge 3/2$, we get $\bar v = \Phi v \in V$ if C_* is sufficiently small.

We next consider the difference estimate. Let $v_1, v_2 \in V$, $\bar v_1 = \Phi v_1$, and $\bar v_2 = \Phi v_2$. Then

$$\|\Phi v_1 - \Phi v_2\|_{W^{1,r}} \le C(\|U\|_{L^3_{wk}} + \|v_1\|_{L^3_{wk}} + \|v_2\|_{L^3_{wk}}) \|v_1 - v_2\|_{W^{1,r}}. \tag{3.5}$$

Taking C_* sufficiently small for $3/2 \le r < 3$, we get $\|\Phi v_1 - \Phi v_2\|_V \le \frac{1}{2} \|v_1 - v_2\|_V$, which shows that Φ is a contraction mapping in V and thus has a unique fixed point. We have proved the unique existence of the solution for (3.3)–(3.4) in V. □

Remark. Since the constant C_r for the Stokes estimate can be taken the same for $r \in [3/2, 3]$, C_2 is independent of r. However, the constant $\tilde{C}_r$ from Lemma 2.1 (ii) blows up as $r \to 3_-$, thus C_* has to shrink to zero as $r \to 3_-$.

Proof of Lemma 3.2. By the difference estimate (3.5), we have

$$\|v_1 - v_2\|_{W^{1,r}} \le C(\|U\|_{L^3_{wk}} + \|v_1\|_{L^3_{wk}} + \|v_2\|_{L^3_{wk}})\|v_1 - v_2\|_{W^{1,r}}.$$

Thus, if $C(\|U\|_{L^3_{wk}} + \|v_1\|_{L^3_{wk}} + \|v_2\|_{L^3_{wk}}) < 1$, we conclude $v_1 = v_2$. □

3.2 Pointwise bound

In this subsection, we will prove pointwise bound of w using $\|w\|_{W^{1,q}} \lesssim C_*$.

For any fixed $x_0 \in B_{1/2} \backslash \{0\}$, let $R = |x_0|/4$ and $E_k = B(x_0, kR)$, $k = 1, 2$.

Note $q^* \in (3, \infty)$. Let s be the dual exponent of q^*, $1/s + 1/q^* = 1$. We have

$$\|w\|_{L^1(E_2)} \lesssim \|w\|_{L^{q^*}(E_2)} \|1\|_{L^s(E_2)} \lesssim C_* R^{4-3/q}.$$

By the interior estimate Lemma 2.2,

$$\|\nabla w\|_{L^{q^*}(E_1)} \lesssim \|f\|_{L^{q^*}(E_2)} + R^{-4+3/q^*} \|w\|_{L^1(E_2)}$$

where $f = U \otimes w + w \otimes (U + w)$. Since $|U| + |w| \lesssim C_*|x|^{-1} \lesssim C_* R^{-1}$ in E_2,

$$\|f\|_{L^{q^*}(E_2)} \lesssim C_* R^{-1} \|w\|_{L^{q^*}(E_2)} \lesssim C_*^2 R^{-1}.$$

We also have $R^{-4+3/q^*} \|w\|_{L^1(E_2)} \lesssim R^{-4+3/q^*} C_* R^{4-3/q} = C_* R^{-1}$. Thus

$$\|\nabla w\|_{L^{q^*}(E_1)} \lesssim C_* R^{-1}.$$

By Gagliardo-Nirenberg inequality in E_1,

$$\|w\|_{L^\infty(E_1)} \lesssim \|w\|_{L^{q^*}(E_1)}^{1-\theta} \|\nabla w\|_{L^{q^*}(E_1)}^{\theta} + R^{-3} \|w\|_{L^1(E_1)},$$

where $1/\infty = (1-\theta)/q^* + \theta(1/q_* - 1/3)$ and thus $\theta = 3/q - 1$. We conclude $\|w\|_{L^\infty(E_1)} \le C_* R^{-\theta}$. Since x_0 is arbitrary, we have proved the pointwise bound, and completed the proof of Theorem 1.1.

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